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Fermions' Tunnelling from the Reissner-Nordström-Anti-de Sitter Black Hole

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Abstract Very recent work of Kerner and Mann involving fermions tunnelling from the Rindler space-time and a general non-rotating black hole is extended to the case of Reissner-Nordström-anti-de Sitter black hole. Due to the couple between the gravity field and electromagnetic field, we introduce the Dirac equation of the charged particles to determine the action of the radiation. We further consider the correction of the thermal spectrum in the unfixed background space time. It is shown that when the energy and charge conservations are considered, the tunnelling rate of fermions is also related to the change of Bekenstein-Hawking entropy, implying the underlying unitary theory is satisfied.

Keywords Hawking radiation · Fermions tunnelling · Reissner-Nordström-anti-de Sitter black hole

1 Introduction

Since Hawking [1] proved that black holes can emit particles from its event horizon in the form of the pure thermal spectrum, Hawking radiation of black holes have been studied extensively by lots of people [2–11]. Among them, the semi-classical method of modeling Hawking radiation as a tunnelling process is most interesting and intriguing. The key point for this method involves the calculation of the imaginary part of the action for emission particles, which in turn is related to the Boltzmann factors for Hawking thermal spectrum. Following this picture, a lot of work until now has been extended to various cases of black holes [12–20] such as charged and rotating space-time, de Sitter and anti-de Sitter black holes as well as some ones in higher dimension [21, 22]. Most of those results strongly manifest that tunnelling provides not only a useful verification of thermodynamic properties of black hole but also an alternate conceptual way for understanding the exact radiation process of black hole.

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However, all their work only involved scalar particles except for the very recent work of Kerner and Mann [23] who studied the tunnelling of spin $1/2$ particle from black hole. Their work, enlightened by the approach of Padmannbhan et al. [14, 24], was mainly inspired by the fact that a black hole should radiate all type of particles and the thermal spectrum therefore should contain particles of all spins. They applied the Dirac Equation to calculate the imaginary part of the action for the Rindler space-time and a general uncharged black hole to serve the semi-classical WKB approximation. Nevertheless, the derived thermal spectrum in their paper is only pure thermal. Actually Parikh and Wilzek [8, 9], considering the self-gravitational interaction and back reaction, have already found that the actual radiation spectrum of the static Schwarzschild and Reissner-Nordström black holes are not pure thermal but has some corrections. So to precisely picture the tunnelling event, the self-gravitational interaction and back reaction of the emitted spin particles should be taken into account.

In this paper, we shall investigate the fermions tunnelling from the Reissner-Nordström-anti-de Sitter black hole in the dynamical background space time. As far as this charged black hole is concerned, the electromagnetic field would couple with the gravity field, the Dirac Equation thus here will be revised as that of charged particles. Then to discuss the correction of the thermal spectrum, the charge conservation also should be incorporated. With regard to the change of angular momentum of the black hole that arises from the spin of the emitted particles, because the total Arnowitt-Deser-Misner (ADM) mass is larger than the Planck mass and the number of the spin up particle statistically is same as its partner's, we ignore the change of the angular momentum of the black hole in our calculation. After considering the energy conservation and charge conservation, we get the corrected spectrum.

The remainder is outlined as follows. In the next section, we will introduce the Dirac Equation of the charged particles to study the pure thermal spectrum of fermions from the Reissner-Nordström-anti-de Sitter black hole. Then in section three, we consider the energy and charge conservations to discuss the correction spectrum. Section four is devoted to our conclusions.

2 The Pure Thermal Spectrum of Fermions from the Reissner-Nordström-Anti-de Sitter Black Hole

The line element of the Reissner-Nordström-anti-de Sitter black hole is

$$ds^2 = f(r)dt_R^2 - f(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{R^2}, \tag{2}$$

in which, M is the black hole mass, Q is the charge parameter, and R is the anti-de Sitter (AdS) radius. The black hole mass relates to its charge Q and horizon radius r_+ by the relation

$$M = \frac{1}{2}\left(r_+ + \frac{r_+^3}{R^2} + \frac{Q^2}{r_+}\right). \tag{3}$$

The electromagnetic vector potential takes the form as

$$A_\mu = \left(-\frac{Q}{r}, 0, 0, 0\right). \tag{4}$$

In fermions tunnelling calculation, it is a good approximation to ignore any change in the angular momentum of the black hole as stated above. For Hawking radiation of spin 1/2 particle from the black hole, when we take the electromagnetic potential of fermions into account additionally, the Dirac Equation reads off

$$i\gamma^{\mu}\left(D_{\mu}-\frac{iqA_{\mu}}{\hbar}\right)\psi+\frac{m}{\hbar}\psi=0, \tag{5}$$

where the Greek indices $\mu, \nu = 0, 1, 2, 3$, q and m are the charge and mass of the fermions and

$$D_{\mu}=\partial_{\mu}+\Omega_{\mu}, \quad \Omega_{\mu}=\frac{1}{2}i\Gamma_{\mu}^{\alpha\beta}\,\Sigma_{\alpha\beta}, \Sigma_{\alpha\beta}=\frac{1}{4}i[\gamma^{\alpha},\gamma^{\beta}]. \tag{6}$$

Based on the relation $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}I$, the Gamma matrix in this background can be chosen as $\gamma^{t}=\gamma^{0}/\sqrt{f}$, $\gamma^{r}=\sqrt{f}\gamma^{3}$, $\gamma^{\theta}=\gamma^{2}/r$ and $\gamma^{\varphi}=\gamma^{3}/r\sin\theta$ while

$$\gamma^{0}=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^{i}=\begin{pmatrix} 0 & \sigma^{i} \\ \sigma^{i} & 0 \end{pmatrix}, \tag{7}$$

in which σ^{i} $(i=1,2,3)$ is the general Pauli Sigma matrix. The matrix for γ^{5} correspondingly takes the form as

$$\gamma^{5}=i\gamma^{t}\gamma^{r}\gamma^{\theta}\gamma^{\varphi}=\frac{i}{r^{2}\sin\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8}$$

In the Dirac field, the solution of the spin up and spin down cases can be expressed as

$$\psi_{\uparrow}(t,r,\theta,\varphi)=\begin{bmatrix} A(t,r,\theta,\varphi)\xi_{\uparrow} \\ B(t,r,\theta,\varphi)\xi_{\uparrow} \end{bmatrix}\exp\left[\frac{i}{\hbar}I_{\uparrow}(t,r,\theta,\varphi)\right], \tag{9}$$

$$\psi_{\downarrow}(t,r,\theta,\varphi)=\begin{bmatrix} C(t,r,\theta,\varphi)\xi_{\downarrow} \\ D(t,r,\theta,\varphi)\xi_{\downarrow} \end{bmatrix}\exp\left[\frac{i}{\hbar}I_{\downarrow}(t,r,\theta,\varphi)\right], \tag{10}$$

where $\xi_{\uparrow/\downarrow}$ is the eigenvectors of σ^{3}, and $I_{\uparrow/\downarrow}$ is the action of the emitted particles with spin up and down. $A(t,r,\theta,\varphi)$ and $B(t,r,\theta,\varphi)$ correspond to the cases of spin up fermions outgoing and ingoing respectively, so do the $C(t,r,\theta,\varphi)$ and $D(t,r,\theta,\varphi)$ for spin-down fermions. In this paper, we proceed to the spin down case for our interest because that its result would obviously be similar to the one of the spin up case. And the other reason of this choose stems from the production of CERN Large Hadronic Collider (LHC) [25], which is proved that more than 70% or 80% of black hole's mass during the life of black holes at the collider is lost during the spin down phase. Inserting (9) into the Dirac Equation (5) and then dividing the exponential term and multiplying $\hbar$, the result of the final expression to leading order in $\hbar$ are

$$\frac{D}{R}\left(\partial_{\theta}I_{\downarrow}-\frac{i}{\sin\theta}\partial_{\varphi}I_{\downarrow}\right)=0, \tag{11}$$

$$\frac{iC}{\sqrt{f(r)}}(\partial_{t}I_{\downarrow}-qA_{t})-D\sqrt{f(r)}\partial_{r}I_{\downarrow}-mC=0, \tag{12}$$

$$\frac{C}{R}\left(\partial_{\theta}I_{\downarrow}-\frac{i}{\sin\theta}\partial_{\varphi}I_{\downarrow}\right)=0, \tag{13}$$

$$\frac{-iD}{\sqrt{f(r)}}(\partial_t I_\downarrow - qA_t) - C\sqrt{f(r)}\partial_r I_\downarrow - mD = 0. \tag{14}$$

With regard to the action $I_\downarrow$, it is not easy to calculate it from above equations. However, taking into account the time-like killing vector $(\frac{\partial}{\partial t})^a$, we can express the action as

$$I_\downarrow = -\omega t + W(r) + J(\theta, \varphi), \tag{15}$$

then we get

$$\frac{D}{R}\left(J_\theta - \frac{i}{\sin\theta} J_\varphi\right) = 0, \tag{16}$$

$$\frac{-iC}{\sqrt{f(r)}}(\omega - \omega_0) - D\sqrt{f(r)}W'(r) - mC = 0, \tag{17}$$

$$\frac{C}{R}\left(J_\theta - \frac{i}{\sin\theta} J_\varphi\right) = 0, \tag{18}$$

$$\frac{iD}{\sqrt{f(r)}}(\omega - \omega_0) - C\sqrt{f(r)}W'(r) - mD = 0, \tag{19}$$

where $\omega_0 = qV_0 = qQ/r_+$. Obviously, (16) and (18) both yield $J_\theta + \frac{i}{\sin\theta} J_\varphi = 0$ regardless the value of C or D, this means that the solutions of J for the outgoing and ingoing cases are the same though $J(\theta, \varphi)$ must be a complex function according to (16) and (18). Hence the contribution of J to the outgoing probability is counteracted by the contribution to the ingoing one in total tunnelling probability. On the other hand, for the case of massless (namely $m = 0$), (17) and (19) have two possible solutions:

$$C = iD, \qquad W'(r) = W'_+(r) = \frac{\omega - \omega_0}{f(r)}, \tag{20}$$

$$C = -iD, \qquad W'(r) = W'_-(r) = -\frac{\omega - \omega_0}{f(r)}, \tag{21}$$

where $W_\pm(r)$ stand for respectively the outgoing and incoming solutions. In regard to the case that $m \neq 0$, (17) and (19) lead to the relation:

$$\left(\frac{C}{D}\right)^2 = \frac{-i(\omega - \omega_0) - \sqrt{f(r)}\,m}{i(\omega - \omega_0) - \sqrt{f(r)}\,m}. \tag{22}$$

Taking the near-horizon approximation, one can finds $C^2 = -D^2$, which is same with the result of the case of massless. Due to the event horizon and infinite red-shift surface of this hole are coincident with each other, thus the geometrical optics limit is reliable and the WKB approximation is applicable. In the semi-classical limit, the tunnelling probability relate to the imaginary part of the emitted particle's action as

$$\Gamma \sim \mathrm{e}^{-2\mathrm{Im} I}. \tag{23}$$

So the overall tunnelling probability of a particle running from inside to outside of the horizon is:

$$\Gamma \sim \frac{P(out)}{P(in)} = \frac{\exp[-2(\mathrm{Im} W_+ + \mathrm{Im} J)]}{\exp[-2(\mathrm{Im} W_- + \mathrm{Im} J)]} = \exp(-4\mathrm{Im} W_+), \tag{24}$$

henceforth we drop the "+" subscript from W, and where

$$W = \int \frac{(\omega - qA_t)}{f(r)}dr = \frac{i\pi(\omega - qA_t)r_+}{r_+ + 3r_+^2/R^2 - Q^2/r_+}. \tag{25}$$

Based on this action, we can easily obtain the Hawking temperature

$$T = \frac{r_+ + 3r_+^2/R^2 - Q^2/r_+}{4\pi r_+}. \tag{26}$$

This is consistent with that in [26]. This is a successful exhibition for the calculation of fermions tunnelling. However, the derived radiation spectrum from (26) is only the pure thermal one because that the background space-time is fixed. Therefore, in order to precisely picture the Hawking radiation, the self-gravitational interaction and back reaction of the emitted spin particles should be taken into account.

3 The Corrected Spectrum of Fermions Tunnelling Radiation

For the Hawking radiation of charged fermions, it also stems from the creation of a pair of virtual particles spontaneously inside the horizon. The positive energy virtual particle tunnels out the horizon and materializes as a real particle escaping to infinity, while the negative energy partner is absorbed by the black hole. This leads to a decrease in the mass of the black hole and the radius of the horizon. When the energy conservation and the charge conservation are taken into account and the total ADM mass together with the charge of the space-time are fixed and allow the to fluctuate, as the particle with a shell of energy ω and charge q runs out, the total mass and charge of the hole would reduce to $M-\omega$ and $Q-q$ accordingly. So the imaginary part of the actual action should be written as

$$\mathrm{Im}W' = \int_{M,Q}^{M-\omega,Q-q} \frac{\pi[-d(M-\omega) + A_t' d(Q-q)]r_+'}{r_+' + 3r_+'^2/R^2 - (Q-q)^2/r_+'}, \tag{27}$$

where $A_t' = -(Q-q)/r$, $r_+' = r_+(M \to M-\omega, Q \to Q-\omega, R)$. To get the last result, one can integrate it directly. But recalling the differential form of the first law of black hole thermodynamics

$$d\omega = TdS + Vdq. \tag{28}$$

Substituting (28) into (27) yields

$$\mathrm{Im}W' = \int_{s_i}^{s_f} \frac{1}{4}ds = \frac{\Delta S}{4}, \tag{29}$$

where r_i is the initial radius corresponding the site of pair-creation inside the event horizon r_+, which is close to the horizon, while r_f is the final radius, which is slightly outside the horizon, and

$$\Delta S = S(r_f) - S(r_i) = \pi r_f^2 - \pi r_i^2, \tag{30}$$

is the change of Bekenstein–Hawking entropy of the black hole before and after the fermions emission.

Expanding ΔS at the near field of r_+ as a Taylor series

$$\Delta S = S(r_f) - S(r_i) = \frac{dS}{dr_+}\Delta r_+ + \frac{1}{2!}\frac{d^2 S}{dr_+^2}(\Delta r_+)^2 + \frac{1}{3!}\frac{d^3 S}{dr_+^3}(\Delta r_+)^3 + \cdots. \tag{31}$$

According to (3), we can change the variant Δr_+ into $(\omega - \omega_0)$, that is,

$$\Delta r_+ = -\frac{2(\omega - \omega_0)}{r_+ + 3r_+^2/R^2 - Q^2/r_+}. \tag{32}$$

Substituting (32) into (31) and invoking the relation of the entropy and the horizon of black hole, $S = \pi r_+^2$, we can get the expansion of ΔS in following form

$$\Delta S = S(r_f) - S(r_i) = -\beta(\omega - \omega_0)\left[1 - \frac{R^2 r_+^2}{3r_+^4 + R^2 r_+^2 - R^2 Q^2}(\omega - \omega_0)\right], \tag{33}$$

where β is the inverse temperature. This result is obviously consistent with an underlying unitary theory. From (33) one can find that the leading term represents the thermal Boltzmann factors $\exp[-\beta(\omega - \omega_0)]$ for the Hawking radiation, and the second term means correction to the tunnelling probability for the response of the unfixed background geometry. That is, after taking into account the energy conservation and the unfixed background space time, the actual radiation spectrum deviates from the pure thermal one.

4 Conclusions

We have investigated Hawking tunnelling radiation of the charged fermions as a tunnelling process from the Reissner-Nordström-anti-de Sitter black hole. In light of the coupling between the matter field and the electromagnetic field, we introduced the Dirac equation of charged particles. Based on the work of Kerner and Mann, we further consider the self-gravitational interaction and back reaction of the emitted particles. This realization is believed to be conceptually clean physically profound. We find the tunnelling probability is related to the change of the entropy, which agrees with the result of the scalar particles and coincides with the underlying unitary theory.

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